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Linear Differential Equations in Infinitely Many Variables.[*]

By William L. Hart.

§ 1. *Introduction.*

In the study of finite systems of ordinary differential equations, linear systems

$$\frac{dx_i}{dt}=\sum_{j=1}^{n} k_{ij}(t)x_j, \qquad (i=1, 2, \ldots, n), \tag{1}$$

are found to have many interesting properties, a number of which are connected with the notion of fundamental sets of solutions of (1). In the present paper the infinite system of ordinary differential equations

$$\frac{dx_i}{dt}=\sum_{j=1}^{\infty} k_{ij}(t)x_j, \qquad (i=1, 2, \ldots), \tag{2}$$

is considered and certain analogous properties are discussed where the notion of fundamental sets of solutions for (2) is defined in an appropriate manner.

Throughout the discussion it is assumed that the $k_{ij}(t)$ in (2) are power series in the complex variable t converging for $|t|\leqq r$. The matrix of elements $(k_{ij}(t))_{i,j=1,2,\ldots}$ is supposed to be of the limited type, in the sense defined by Hilbert. The set of complex numbers $\xi \mathrel{\overline{\overline{\cdot}}} (x_1, x_2, \ldots)$ is supposed to represent a point in Hilbert complex space; *i. e.*, if $\bar{x}_i$ represents the conjugate to x_i, then $\Sigma_{i=1}^{\infty}\, \bar{x}_i x_i$ converges.

In § 2 the system of notation used in the paper is explained. There is then given a brief outline of the results from the theory of limited linear and bi-linear forms in infinitely many variables which are used in the later work. In § 3 the homogeneous system (2) is considered. Under certain assumptions as to the limited character of the matrix $(k_{ij}(t))$ the existence of an analytic solution $(x_1(t), x_2(t), \ldots)$ taking on, for $t=t_0$, a given set of values $(a_1, a_2, \ldots)$, is established and an analytical representation for it obtained in terms of an auxiliary matrix.

In § 4 the adjoint system to the equations (2),

$$\frac{dx_i}{dt}=-\sum_{j=1}^{\infty} k_{ji}(t)x_j, \tag{3}$$

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is considered. As a result of the conditions satisfied by system (2), the system (3) is of the same type as (2). It is found that, if $\xi_1(t) = (x_{11}(t), x_{21}(t), \ldots.)$ and $\xi_2(t) = (x_{12}(t), x_{22}(t), \ldots.)$ are solutions of (2) and (3), respectively, then $\Sigma_{i=1}^{\infty} x_{i1}(t) x_{i2}(t)$ is a constant.

In § 5 the notion of fundamental sets of solutions of system (2) is defined in a manner suggested by that followed in the finite case if, in the latter, where there enters the idea of the non-vanishing of the determinant of the fundamental set, there is put the concept of a limited matrix possessing a unique limited reciprocal matrix. The existence of fundamental sets is established and certain of their properties are discussed. In this connection the adjoint system (3) proves to be useful because the reciprocal matrix of a fundamental set of solutions of (2) is a matrix of solutions of the system (3).

In § 6 the non-homogeneous system is considered which results from (2) on adding to the *i-th* equation the term $k_{i0}(t)$. The general solution of the modified system is obtained as the sum of a particular solution and the general solution of the system (2). On using a method analogous to that of variation of parameters, a particular solution is obtained by a quadrature with the aid of a fundamental set of solutions of (2).

Infinite systems of differential equations have been considered by H. Von Koch,* F. R. Moulton,† E. H. Moore,‡ J. F. Ritt,§ T. H. Hildebrandt,|| and the author.¶ The general systems of Von Koch, Moulton and the author do not include the system (2) of the present paper. J. F. Ritt considered a single differential equation of infinite order which is a problem of a different type. Moore and Hildebrandt considered problems in general analysis including as special instances a certain infinite system of the type (2), in the field of reals. The results of Moore were not of the nature of those obtained in the present paper. The results of Hildebrandt generalize the theorems on finite systems in such a way that the notion of the determinant of the fundamental sets of solutions is retained, whereas in the present paper the matrices of the fundamental sets need not possess determinants.

§ 2. *Limited Bi-Linear Forms.*

In the work of the present paper it will be convenient to adopt the system of notation which is explained in the following paragraphs. The same type of

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* *Öfversigt af Kongliga Vetenskaos Akademiens Förhandlingar*, Vol. LVI (1899), pp. 395–411.

† *Proceedings of the National Academy of Sciences*, Vol. I (1915), p. 350.

‡ *Atti dei IV Congresso Internatzionale dei Matematici*, Vol. II (Roma, 1908), p. 98.

§ *Transactions of the American Mathematical Society*, Vol. XVIII (1917), p. 27.

|| *Ibid.*, Vol. XVIII (1917), p. 73.

¶ *Ibid.*, Vol. XVIII (1917), p. 125.

notation has long been used by E. H. Moore[*] in his work on Integral Equations. In its general form the notation is like that met in vector and matrix algebra.

An infinite set of values $(x_1, x_2, \ldots.)$ will be represented by a single Greek letter, for example ξ. Then, the point, or vector, ξ in the space of infinitely many dimensions is considered as a function of the variable i whose range is $I = (1, 2, \ldots.)$; that is, $\xi(i) = x_i$. Points $(x_1, x_2, \ldots.)$ will be represented either by ξ, η, γ or α, with perhaps an added notation.

An infinite matrix $(k_{ij})_{i,j=1,2,\ldots.}$ in which i denotes the row, j the column, will be represented by a single Greek letter such as κ. Then κ is considered as a function of the two indices i and j for which $\kappa(i, j) = k_{ij}$. Any Greek letter except ξ, η, γ or α will in the future represent a matrix.

A Greek letter with an added notation represents the point or matrix obtained by adding the given notation to each element of the point or matrix given by the unannotated Greek letter. For example,

$$\xi' = (x_1', x_2', \ldots.), \qquad \xi(t) = \{x_1(t), x_2(t), \ldots.\},$$
$$\kappa(t) = (k_{ij}(t))_{i,j=1,2,\ldots.}, \qquad \kappa_n = (k_{ijn})_{i,j=1,2,\ldots.},$$
$$\frac{d\xi}{dt} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots.\right), \qquad \frac{d\kappa}{dt} = \left(\frac{dk_{ij}}{dt}\right)_{i,j=1,2,\ldots.}.$$

In the subsequent work infinite series enter to a great extent. On using what is seen to be analogous to vector notation, let there be introduced the abbreviation

$$\sum_{i=1}^{\infty} x_i y_i = S\xi\eta, \qquad [\xi = (x_1, x_2, \ldots.), \ \eta = (y_1, y_2, \ldots.)].$$

The S is a summation operation giving what may be called the scalar product of the two points or vectors ξ and η. In similar fashion, the point η whose coordinates y_i are given by $y_i = \Sigma_{j=1}^{\infty} k_{ij} x_j$ and the point α whose coordinates z_i are given by $z_i = \Sigma_{j=1}^{\infty} x_j k_{ji}$, will be represented by, respectively,

$$\eta = S\kappa\xi, \quad \alpha = S\xi\kappa.$$

The operation S thus produces a point from a matrix and a point. Let κ' and κ'' be two matrices; the product matrix λ is defined by

$$\lambda = \Big(\sum_{g=1}^{\infty} k_{ig}' k_{gj}''\Big)_{i,j=1,2,\ldots.} = S\kappa'\kappa'',$$

or, S produces a matrix from two given matrices. The meaning of such expressions as $S\xi S\kappa\eta$, $S\kappa S\kappa'\kappa''$, $S(S\kappa\xi)\eta$, is then evident. The expressions $S^2\xi\kappa\eta$, $S^2\kappa'\kappa\xi$ are defined by, respectively,

$$S^2\xi\kappa\eta = \sum_{i,j=1}^{\infty} x_i k_{ij} y_j = \lim_{n=\infty} \sum_{i,j=1}^{n} x_i k_{ij} y_j,$$
$$S^2\kappa'\kappa\xi = \Big(\sum_{i,j=1}^{\infty} k_{1i}' k_{ij} x_j, \ \sum_{i,j=1}^{\infty} k_{2i}' k_{ij} x_j, \ \ldots.\Big).$$

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[*] *Proceedings of the Fifth International Congress* (Cambridge), Vol. I, p. 230.

In the following discussion, "$\Sigma_{i,j=1}^{\infty}$" will always mean the limit indicated in the first of the expressions above.

It will sometimes be convenient to use the expressions considered in the preceeding paragraph when the sums involved run only from 1 to m. Then write S_m instead of S:

$$S_m\xi\eta = \sum_{i=1}^{m} x_i y_i, \quad S_m x\xi = \left(\sum_{i=1}^{m} k_{1i}x_i, \sum_{i=1}^{m} k_{2i}x_i, \ldots\ldots\right), \text{ etc.}$$

A stroke over a letter (*e. g.*, $\bar{x}$, $\bar{\xi}$) will always represent the conjugate complex number to the quantity bearing the notation.

DEFINITION 1. *A point ξ belongs to Hilbert complex space in case there exists*

$$S\bar{\xi}\xi = \sum_{i=1}^{\infty} \bar{x}_i x_i, \qquad [\xi = (x_1, x_2, \ldots\ldots)].$$

The number $\sqrt{S\bar{\xi}\xi}$ will be called the modulus of ξ and will be denoted by $M\xi$.

The proofs of certain of the properties of limited linear and bi-linear forms in infinitely many variables listed below are found in a paper[*] by E. Hellinger and O. Toeplitz. In the future the notation "*H. T. page* —" refers to a page in this article. Hellinger and Toeplitz in most cases give the detailed proofs only for the case of real values of the variables, but as they remark (H. T. page 305), the extension to complex values is a simple matter.

PROPERTY 1 (*H. T. page* 295). *Suppose there exists a number l such that, for all points $\eta = (y_1, y_2, \ldots\ldots)$ of Hilbert space there holds, for every m,*

$$|S_m\xi\eta| \leqq lM\eta.$$

Then it follows that ξ belongs to Hilbert space and that $M\xi \leqq l$.

PROPERTY 2. *Let $\xi(u)$ and $\eta(u)$, where u is a complex variable belonging to a region R in the u-plane, belong to Hilbert space for every u. Suppose that*

$$S\bar{\xi}(u)\xi(u) \leqq l$$

for all u, while $S\bar{\eta}(u)\eta(u)$ converges uniformly for all u. Then

$$S\xi(u)\eta(u) = \sum_{i=1}^{\infty} x_i(u)y_i(u)$$

converges absolutely uniformly for all u of R.

This property is established with the aid of the Schwarz[†] inequality. For,

$$\sum_{i=n+1}^{\infty} |x_i(u)| \cdot |y_i(u)| \leqq \sqrt{\sum_{i=n+1}^{\infty} \bar{x}_i(u)x_i(u)}\sqrt{\sum_{i=n+1}^{\infty} \bar{y}_i(u)y_i(u)} \leqq l\sqrt{\sum_{i=n+1}^{\infty} \bar{y}_i(u)y_i(u)},$$

which approaches zero uniformly for all u as $n = \infty$.

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[*] *Mathematische Annalen*, Vol. LXIX (1910), p. 289.

[†] H. T. page 293.

DEFINITION 2. *A matrix* $\varkappa=(k_{ij})_{i,j=1,2,\dots}$ *is limited* l *in case for every pair of Hilbert points* ξ *and* η *and for every* m,

$$|S_mS_m\xi\varkappa\eta|=|\sum_{i=1}^{m}\sum_{j=1}^{m}x_ik_{ij}y_j|\leqq lM\xi M\eta.$$

If $\varkappa$ is an infinite matrix, the conjugate matrix $\bar{\varkappa}$ and the transposed matrix $\varkappa'$ are defined by

$$\bar{\varkappa}=(\bar{k}_{ij})_{i,j=1,2,\dots},\qquad \varkappa'=(k_{ji})_{i,j=1,2,\dots}.$$

PROPERTY 3 (*H. T. page* 310). *Suppose that* $\varkappa$ *is limited* l. *Then* $\varkappa'$ *and* $\bar{\varkappa}$ *are also limited* l.

PROPERTY 4 (*H. T. page* 297). *Let* $\varkappa$ *be limited* l. *Then each row and each column of* $\varkappa$ *constitutes a point, belonging to Hilbert space, whose modulus is at most* l.

PROPERTY 5 (*H. T. page* 299). *If* $\varkappa$ *is limited* l *then, for every pair of Hilbert points* ξ *and* η, *the following infinite sums are defined and equal:*

$$S\xi S\varkappa\eta,\quad S(S\xi\varkappa)\eta,\quad S^2\xi\varkappa\eta.$$

PROPERTY 6 (*H. T. page* 300). *Let* $\varkappa'$ *and* $\varkappa''$ *be limited* l' *and* l'', *respectively. Then the product matrix* $\lambda=S\varkappa'\varkappa''$ *is limited* $l'l''$.

DEFINITION 3. *Let* $\varkappa$ *and* λ *be two limited matrices. The matrix* λ *is a right-hand, or a left-hand reciprocal of* $\varkappa$ *according as*

$$S\varkappa\lambda=\delta \quad or \quad S\lambda\varkappa=\delta,$$

where δ *is the unit matrix* $(d_{ij})_{i,j=1,2,\dots}$, $d_{ii}=1$, $d_{ij}=0\,(i\neq j)$.

PROPERTY 7 (*H. T. page* 311). *If the matrix* $\varkappa$ *is limited and possesses both right and left-hand reciprocals, it follows that there exists uniquely a right-hand and a left-hand reciprocal, and moreover, they are equal to each other.*

PROPERTY 8 (*H. T. page* 312). *Let* $\varkappa$ *be a limited matrix and suppose that there exists a unique right* (*left*) *hand reciprocal* λ. *Then* λ *is also the unique left* (*right*) *hand reciprocal of* $\varkappa$.

In case $\varkappa$ possesses a unique left-hand and right-hand reciprocal λ, this matrix will be called simply the reciprocal of $\varkappa$. By direct verification it is established that there holds.

PROPERTY 9. *Suppose that* $\varkappa$ *is limited and possesses a reciprocal* λ. *Then for every two points* ξ *and* η *of Hilbert space, the two infinite systems of linear equations*

$$\xi=S\varkappa\eta,\quad \eta=S\lambda\xi$$

are equivalent.

§ 3. *The Homogeneous System.*

The subject for consideration is the system of equations (2) which in the notation of § 2 becomes

$$\frac{d\xi}{dt} = S\varkappa(t)\xi, \tag{4}$$

where $\varkappa(t) = (k_{ij}(t))_{i,j=1,2,\ldots}$ is a matrix each of whose elements is a power series in the complex variable t converging for $|t| \leqq r$. Throughout the discussion it will be assumed that $\varkappa(t)$ satisfies the hypothesis

(H_1). *The coefficients* $\varkappa_n = (k_{ijn})_{i,j=1,2,\ldots}$ *in the matrix equation*

$$\varkappa(t) = \sum_{n=0}^{\infty} \varkappa_n t^n,$$

are such that $\varkappa_n$ *is limited* l_n, *and moreover,*

$$\sum_{n=0}^{\infty} l_n r^n$$

converges. Therefore there exists a number N *such that* $l_n \leqq N r^n$.

PROPOSITION 1. *The matrix* $\varkappa(t)$ *is limited for* $|t| \leqq r$ *and its limit may be taken as the quantity* $l(t)$ *defined by*

$$l(t) = \sum_{n=0}^{\infty} l_n |t|^n.$$

To establish this result, recall the definition of a limited matrix. Let ξ and η be two points of Hilbert space. Then it follows that

$$|S_m S_m \xi \varkappa(t) \eta| = |S_m S_m \xi (\sum_{n=0}^{\infty} \varkappa_n t^n) \eta| = |\sum_{n=0}^{\infty} t^n (S_m S_m \xi \varkappa_n \eta|,$$

$$\leqq M\xi M\eta (\sum_{n=0}^{\infty} l_n |t|^n) = l(t)\,(M\xi M\eta). \tag{5}$$

It is seen that (5) states the limited character of $\varkappa(t)$ and that $l(t)$ may be taken as the limit.

PROPOSITION 2. *For every* i *the series*

$$\sum_{g=1}^{\infty} \bar{k}_{ig}(t) k_{ig}(t) \tag{6}$$

converges uniformly for all $|t| \leqq r$. *Similarly, for every* g *the same condition is satisfied by the series*

$$\sum_{i=1}^{\infty} \bar{k}_{ig}(t) k_{ig}(t). \tag{7}$$

From the fact that $k_{ig}(t) = \sum_{n=0}^{\infty} k_{ign} t^n$ it is seen that

$$\sum_{g=m_1}^{m_2} \bar{k}_{ig}(t) k_{ig}(t) = \sum_{n,h=0}^{\infty} \bar{t}^n t^h (\sum_{g=m_1}^{m_2} \bar{k}_{ign} k_{igh}) \ll \sum_{n,h=0}^{\infty} l_h l_n \bar{t}^n t^h, \tag{8}$$

where the dominance relation is a consequence of Property (4) in view of the assumption of (H_1) as to the matrices $\varkappa_n$. As a result of (8) it may be stated that for every assigned number $l > 0$ an integer n_l can be determined so that for all m_1 and m_2 and for $|t| \leqq r$,

$$\Big| \sum_{n,\,h=n_l}^{\infty} \bar{t}^n t^h \Big(\sum_{g=m_1}^{m_2} \bar{k}_{ign} k_{igh} \Big) \Big| \leqq \frac{l}{2}. \tag{9}$$

Because of the convergence for all n and h of the series $\Sigma_{g=1}^{\infty} \bar{k}_{ign} k_{igh}$, an integer m_l can be chosen so large that, for $m_1 \geqq m_l$, $m_2 \geqq m_l$, and for $|t| \leqq r$,

$$\Big| \sum_{n,\,h=0}^{n_l} \bar{t}^n t^h \Big(\sum_{g=m_1}^{m_2} \bar{k}_{ign} k_{igh} \Big) \Big| \leqq \frac{l}{2}. \tag{10}$$

From (9) and (10) it follows that the series in (8) is in absolute value at most l for $|t| \leqq r$, which establishes the proposition.

PROPOSITION 3. *Let t_0 be a point in the region $|t| \leqq r$. If $\lambda(t-t_0)$ represents the matrix obtained from $\varkappa(t)$ by expanding each element of $\varkappa(t)$ as a series in $(t-t_0)$, then the coefficient matrices λ_n in*

$$\lambda(s) = \sum_{n=0}^{\infty} \lambda_n s^n, \qquad (s = t - t_0),$$

are limited by numbers l'_n, and $\Sigma_{n=0}^{\infty} l'_n r_0^n$ converges, where r_0 is any value satisfying $r_0 < r - |t_0|$.

In other words, Proposition (3) states that the analytic continuation of $\varkappa(t)$ to the point t_0 satisfies, for $|s| \leqq r_0$, conditions which are precisely similar to those satisfied by $\varkappa(t)$ for $|t| \leqq r$.

Let the notation for the analytic continuations of the elements of $\varkappa(t)$ be

$$k_{ij}(t) = \sum_{n=0}^{\infty} h_{ijn} s^n,$$

so that $\lambda(s) = \Sigma_{n=0}^{\infty} \lambda_n s^n$, where $\lambda_n = (h_{ijn})_{i,\,j=1,\,2,\,\ldots\ldots}$. In order to show that the λ_n are limited and that the limits satisfy the convergence condition of the proposition, recall Definition 2. Let ξ and η be two points of Hilbert space. Then

$$S_m S_m \xi \varkappa(t) \eta = \sum_{n=0}^{\infty} C_n t^n, \qquad (C_n = S_m S_m \xi \varkappa_n \eta). \tag{11}$$

From (H_1) it follows that

$$|C_n| \leqq l_n M\xi M\eta,$$

and, hence,

$$\sum_{n=0}^{\infty} C_n t^n \ll (M\xi M\eta) \sum_{n=0}^{\infty} l_n t^n, \tag{12}$$

where the right side converges for $|t| \leqq r$. Consequently, if l'_n represents the coefficient of s^n in the series formed by continuing the series on the right side of (12) to the point $t = |t_0|$, the analytic continuation of the series on the left to the point $t = t_0$ satisfies

$$\sum_{n=0}^{\infty} C_n t^n = \sum_{n=0}^{\infty} B_n s^n \ll \Big(\sum_{n=0}^{\infty} l'_n s^n\Big) M\xi M\eta, \qquad (|s| < r - |t_0|). \tag{13}$$

But, $B_n = S_m S_m \xi \lambda_n \eta$, and, hence, it has been established that

$$|S_m S_m \xi \lambda_n \eta| \leqq l'_n M\xi M\eta.$$

From (13) it is seen that $\Sigma_{n=0}^{\infty} l'_n r_0^n$ converges if $r_0 < r - |t_0|$.

As a consequence of Proposition (3) the theorems stated below, where the hypotheses refer to the initial point $t=0$, apply to the case $t=t_0$, if r is changed to r_0 and l_n to l'_n.

§ 4. *The Existence Theorem for the System* (4).

The fundamental theorem concerning the solution of (4) involves functions $\xi(t)$ of a type which it will be convenient to represent by the class $\mathfrak{M}$.

DEFINITION (4). *A function $\xi(t)$ belongs to the class $\mathfrak{M}$ in case there exists a $w (0 < w \leqq r)$ so that, for $|t| \leqq w$, $\xi(t)$ belongs to Hilbert space, and if the coordinates $x_i(t)$ of $\xi(t)$ are convergent power series, $x_i(t) = \Sigma_{n=0}^{\infty} x_{in} t^n$, or*

$$\xi(t) = \sum_{n=0}^{\infty} \xi_n t^n, \qquad [\xi_n = (x_{1n}, x_{2n}, \ldots .)]. \tag{14}$$

Moreover, it is possible to interchange infinite summations in the following expression and to write

$$S\varkappa(t)\xi(t) = S\Big(\sum_{n=0}^{\infty} \varkappa_n t^n\Big)\Big(\sum_{m=0}^{\infty} \xi_m t^m\Big) = \sum_{n,\,m}^{0,\,\infty} t^{n+m} S\varkappa_n \xi_m. \tag{15}$$

A function $\xi(t)$ will be said to belong to the class $\mathfrak{M}$ at t_1 if it satisfies conditions corresponding to those just enumerated with $\varkappa(t)$ replaced by its analytic continuation to the point t_1.

PROPOSITION 4. *Suppose that $\xi(t)$ is a function satisfying* (14) *and that, instead of* (15), *there holds*

$$M\xi(t) \leqq K < \infty, \qquad (|t| \leqq w). \tag{16}$$

Then it follows that $\xi(t)$ satisfies the condition (15).

Let the proposition be established with the aid of Property (2). Identify the $\xi(u)$ and $\eta(u)$ of that result with $\xi(t)$ and the point $(k_{i1}(t), k_{i2}(t), \ldots .)$, respectively. In view of Proposition (2) and since $M\xi(t) \leqq K$, it follows that, as a consequence of Property (2), the series

$$\sum_{j=1}^{\infty} k_{ij}(t) x_j(t) = \sum_{j=1}^{\infty} \Big(\sum_{n=0}^{\infty} k_{ijn} t^n\Big)\Big(\sum_{m=0}^{\infty} x_{jm} t^m\Big)$$

converges uniformly. Therefore it may be rearranged as a power series in t; in other words $\xi(t)$ satisfies (15).

The existence of a solution of (4) taking on a given set of values for $t=0$ is established in

THEOREM I. *Let the system* (4) *satisfy* (H_1) *and suppose that* ξ_0 *is a point in Hilbert space. Then, among all functions of the class* $\mathfrak{M}$ *there exists uniquely for* $|t|<r$, *a function* $\xi(t)$ *which is a solution of* (4) *satisfying* $\xi(0)=\xi_0$. *Furthermore, the coefficients* ξ_n *in the expansion of* $\xi(t)$,

$$\xi(t)=\sum_{n=0}^{\infty}\xi_n t^n,$$

are given by $\xi_n=S\lambda_n\xi_0$ *where the matrices* λ_n *are the coefficients in the expansion of a certain matrix* $\lambda(t)=\Sigma_{n=0}^{\infty}\lambda_n t^n$ *satisfying the following conditions for* $|t|<r$:

The matrices λ_n *are limited* b_n *and the series*

$$l'(t)=\sum_{n=0}^{\infty}b_n|t|^n \tag{17}$$

converges for $|t|<r$.

Moreover, $\xi(t)$ *can be written in the form*

$$\xi(t)=S\lambda(t)\xi_0,$$

and the series giving the respective components $x_i(t)$ *of* $\xi(t)$ *converge in* t, *uniformly with respect to the index* i.

The theorem will be established by the method of undetermined coefficients. Suppose there exists a solution $\xi(t)$ of (4) belonging to the class $\mathfrak{M}$ and satisfying $\xi(0)=\xi_0$. Then it follows that

$$\xi(t)=\sum_{n=0}^{\infty}\xi_n t^n,\qquad \frac{d\xi(t)}{dt}=\sum_{n=1}^{\infty}n\xi_n t^{n-1}.$$

Hence there is obtained

$$\begin{aligned}\sum_{n=1}^{\infty}n\xi_n t^{n-1}&=S\varkappa(t)\Big(\sum_{n=0}^{\infty}\xi_n t^n\Big)=S\Big(\sum_{n=0}^{\infty}\varkappa_n t^n\Big)\Big(\sum_{m=0}^{\infty}\xi_m t^m\Big)\\&=\sum_{n,\,m=0}^{\infty}(S\varkappa_n\xi_m)t^{n+m}=\sum_{g=0}^{\infty}C_g t^g,\qquad \Big(C_g=\sum_{\substack{n,\,m=0\\ n+m=g}}^{g}S\varkappa_n\xi_m\Big),\end{aligned}\tag{18}$$

where the interchange of S and the infinite summations with respect to m and n is seen to be legitimate because $\xi(t)$ belongs to the class $\mathfrak{M}$. The equation (18) serves to determine uniquely the coefficients ξ_n; there results

$$\xi_0=\xi_0,\quad n\xi_n=C_{n-1},\qquad (n=1,2,\ldots.), \tag{19}$$

which gives

$$\left.\begin{array}{ll}
\xi_1 = S\varkappa_0\xi_0 = S\lambda_1\xi_0 , & (\lambda_1 = \varkappa_0) , \\
2\xi_2 = S\varkappa_1\xi_0 + S^2\varkappa_0\lambda_1\xi_0 = 2S\lambda_2\xi_0 , & \left(\lambda_2 = \dfrac{\varkappa_1 + S\varkappa_0\lambda_1}{2}\right) , \\
\dots\dots\dots\dots\dots\dots\dots\dots , & \dots\dots\dots\dots\dots , \\
n\xi_n = S\varkappa_{n-1}\xi_0 + \dots\dots & \\
\qquad + S\varkappa_0\xi_{n-1} = nS\lambda_n\xi_0 , & \left(\lambda_n = \dfrac{\varkappa_{n-1} + S\varkappa_{n-2}\lambda_1 + \dots\dots + S\varkappa_0\lambda_{n-1}}{n}\right) .
\end{array}\right\} \quad (20)$$

In order to prove the theorem it will first be shown that the series giving the components $x_i(t)$ of the function $\xi(t) = \Sigma_{n=0}^{\infty}\xi_n t^n$, resulting from the coefficients ξ_n found in (20), converge absolutely for $|t| < r$ and uniformly with respect to the index i.

Recall that the matrix $\varkappa_n$ is limited $l_n \leqq N/r^n$; in the work below it will be assumed that the limit is equal to N/r^n. Then, because of Definition 2, it follows that

$$|\xi_1| \leqq NM\xi_0 = b_1 M\xi_0 , \qquad (b_1 = N) .$$

From Property (6) of limited matrices it is seen that λ_2 is limited b_2,

$$b_2 = \frac{N}{2}\left(N + \frac{1}{r}\right) ,$$

and hence $|\xi_2| \leqq b_2 M\xi_0$. On proceeding in this manner it can be established that every matrix λ_n is limited b_n, where

$$b_n \ = \frac{1}{n}\left(\frac{N}{r^{n-1}} + b_1\frac{N}{r^{n-2}} + \dots\dots + b_{n-1}N\right) ,$$

$$b_{n+1} = \frac{1}{n+1}\left(\frac{N}{r^n} + b_1\frac{N}{r^{n-1}} + \dots\dots + b_{n-1}\frac{N}{r} + b_n N\right) .$$

It follows that there is the relation

$$b_{n+1} = \frac{b_n}{n+1}\left(\frac{n}{r} + N\right) ,$$

and that $|\xi_n| \leqq b_n M\xi_0$.

LEMMA 1. *The series* $\Sigma_{n=0}^{\infty} b_n t^n$ *converges for* $|t| < r$.

To see this consider the absolute value q_n of the ratio of the *n-th* to the $(n-1)$*-th* term:

$$q_n = \frac{b_n}{b_{n-1}}|t| = \left(\frac{n-1}{n}\right)\left(\frac{N_r}{n-1} + 1\right)\left|\frac{t}{r}\right| .$$

Since for $|t| < r$ $\lim_{n=\infty} q_n < 1$, the lemma is established. Moreover, since

$$\xi(t) = \sum_{n=0}^{\infty}\xi_n t^n = \sum_{n=0}^{\infty}(S\lambda_n\xi_0)t^n \ll \xi_0 + M\xi_0\Big(\sum_{n=1}^{\infty} b_n t^n\Big) , \qquad (\lambda_0 = 1) , \qquad (21)$$

it follows that the series for each component $x_i(t)$ of $\xi(t)$ converges just as fast and in the same region as the series in Lemma 1.

Now define a matrix $\lambda(t)$ formally by the equation

$$\lambda(t)=\sum_{n=0}^{\infty}\lambda_n t^n, \qquad \{\lambda(t)=(m_{ij}(t))_{i,j=1,2,\ldots}\}. \tag{22}$$

The series in (22) (one for each component of $\lambda(t)$) converge, because of Lemma (1), since $|\lambda_n|\leqq b_n$. The matrix $\lambda(t)$ therefore satisfies conditions corresponding to those imposed on $\varkappa(t)$ in hypothesis (H_1). Hence $\lambda(t)$ is limited $l'(t)$ where $l'(t)$ has the definition (17). From the fact that $\lambda(t)$ is limited uniformly for $|t|\leqq r_1<r$, it follows that $S\lambda(t)\xi_0$ converges uniformly and, therefore,

$$S\lambda(t)\xi_0=S(\sum_{n=0}^{\infty}\lambda_n t^n)\xi_0=\sum_{n=0}^{\infty}(S\lambda_n\xi_0)t^n=\xi(t).$$

To complete the proof of the theorem it remains to show that $\xi(t)$ satisfies (15). First consider

COROLLARY (1). *The modulus of the function $\xi(t)$ satisfies the equation*

$$M\xi(t)=\sqrt{S\bar{\xi}(t)\xi(t)}\leqq l'(t)M\xi_0.$$

It has been seen that $\xi(t)=S\lambda(t)\xi_0$. Hence, if α is a point in Hilbert space, it follows from Definition (2) that, since $\lambda(t)$ is limited $l'(t)$,

$$|S\alpha\xi(t)|=|S\alpha(S\lambda(t)\xi_0)|\leqq M\alpha(l'(t)M\xi_0).$$

Hence, in view of Property (1) it follows that $M\xi(t)$ has the upper bound stated in the corollary.

It is now easily verified that $\xi(t)$ satisfies the hypothesis of Proposition (4) and therefore belongs to the class $\mathfrak{M}$. This completes the proof of the theorem.

COROLLARY (2). *The solution $\xi(t)$ has the property that the two series*

$$M\xi(t)=\sum_{i=1}^{\infty}\bar{x}_i(t)x_i(t), \quad M\left(\frac{d\xi(t)}{dt}\right)=\sum_{i=1}^{\infty}\overline{\frac{dx_i(t)}{dt}}\,\frac{dx_i(t)}{dt}, \tag{23}$$

converge uniformly for $|t|\leqq r_1$ for all $r_1<r$.

There is obtained

$$\begin{aligned}|\sum_{i=n}^{m}\bar{x}_i(t)x_i(t)|&=|\sum_{i=n}^{m}(\sum_{h,j=0}^{\infty}\bar{x}_{ij}\bar{t}^j x_{ih}t^h)|\\ &\leqq\sum_{h,j=0}^{\infty}|t|^{h+j}|\sum_{i=n}^{m}\bar{x}_{ij}x_{ih}|\ll(M\xi_0)^2(\sum_{h,j=0}^{\infty}b_hb_j|t|^{h+j}),\end{aligned}$$

where the dominance relation is a consequence of the fact that $\xi_n=S\lambda_n\xi_0$. Then, to prove the uniform convergence of $S\bar{\xi}(t)\xi(t)$ one would proceed from this point on precisely as in the work leading to (9) and (10) of Proposition (2).

The proof of the uniform convergence of the other series of the corollary is of similar nature.

The corollary to Theorem I and the Theorem II which follow are of importance in the consideration of fundamental sets of solutions.

COROLLARY (3). *Let $\theta(t)$ be a matrix $(x_{ij}(t))$ in which every column constitutes a solution of* (4). *Then, if all these solutions belong to the class $\mathfrak{M}$, the matrix satisfies the equation*

$$\theta(t)=S\lambda(t)\theta(0). \tag{24}$$

The equation (24) is an obvious result of the condition which holds for every column of $\theta(t)$.

THEOREM II. *Let $\theta(t)$ be a matrix in which each column constitutes a solution of* (4) *belonging to the class $\mathfrak{M}$. Suppose that $\theta(0)$ is limited by the quantity $L(0)$. Then $\theta(t)$ is limited $L(t)$,*

$$L(t)=L(0)l'(t). \tag{25}$$

To establish this result refer to Corollary (3) to Theorem I. It is seen from Property (6) of § 2 that, as a consequence of equation (24), $\theta(t)$ is limited by the product of the limits of $\lambda(t)$ and $\theta(0)$, which is the result stated in the theorem.

§ 4. *The Adjoint System.*

In the consideration of a finite system of linear differential equations the solutions of the adjoint system are found to have certain interesting properties in relation to the solution of the given system. On carrying over to infinite systems the classical definition of adjoint systems, there is obtained

DEFINITION 5. *The system of linear differential equations adjoint to the system* (4) *is the system*

$$\frac{d\xi}{dt}=-S\varkappa'(t)\xi, \tag{26}$$

where $\varkappa'(t)$ is the transposed matrix of $\varkappa(t)$.

The hypothesis (H_1) was symmetrical in its conditions on the rows and columns of $\varkappa(t)$ so that, as a consequence, system (26) is of the same type as (4). A useful relation between the solutions of (4) and (26) is

THEOREM III. *Let $\xi_1(t)$ and $\xi_2(t)$ be two functions belonging to the class $\mathfrak{M}$ for $|t|\leqq r_1<r$ which are, respectively, solutions of the equations* (4) *and* (26). *Then*

$$\frac{d}{dt}\{S\xi_1(t)\xi_2(t)\}=0.$$

On formally differentiating $S\xi_1\xi_2$, there is obtained

$$\frac{d}{dt}\{S\xi_1(t)\xi_2(t)\}=S\left(\frac{d\xi_1}{dt}\right)\xi_2+S\xi_1\left(\frac{d\xi_2}{dt}\right). \tag{27}$$

Assume for the present that the term by term differentiation is legitimate. Then, on substituting the values of $d\xi_1/dt$ and $d\xi_2/dt$, it follows that

$$\begin{aligned}\frac{d}{dt}\{S\xi_1(t)\xi_2(t)\} &=S\{S\varkappa(t)\xi_1(t)\}\xi_2(t)-S\xi_1(t)\{S\varkappa'(t)\xi_2(t)\}\\ &=S^2\xi_2(t)\varkappa(t)\xi_1(t)-S\xi_1(t)\{S\xi_2(t)\varkappa(t)\}\\ &=S^2\xi_2(t)\varkappa(t)\xi_1(t)-S^2\xi_2(t)\varkappa(t)\xi_1(t)=0,\end{aligned}$$

where the changes in the forms of expressions are justified by Property (5) of § 2.

To justify the term by term differentiation in (27), recall Property (2) and consider proving that the infinite series on the right in (27) converge uniformly. For the first term, identify the $\eta(u)$ and $\xi(u)$ of Property (2) with $\xi_2(t)$ and $d\xi_1/dt$, respectively. The hypothesis concerning η is satisfied because of Corollary (2) to Theorem I. Since

$$\frac{d\xi_1(t)}{dt}=S\varkappa(t)\xi_1(t)=S\varkappa(t)\{S\lambda(t)\xi_0\}, \qquad (\xi_1(0)=\xi_0),$$

it follows that the modulus of $d\xi_1/dt$ is at most $l(t)l'(t)M\xi_0$ which is uniformly bounded for $|t|\leqq r_1$. Hence the first term on the right in (27) converges uniformly and, similarly, it is established that the second term has the same property. Consequently, the term by term differentiation leading to (27) was legitimate and the proof of the theorem is complete.

§ 5. *Fundamental Sets of Solutions.*

In considering finite systems of linear differential equations the notion of fundamental sets of solutions is of great importance. In the definition of such sets, the determinant of the matrix formed by the set plays a primary rôle. On replacing the idea of the non-vanishing of this determinant by the notion of a limited matrix possessing a unique limited reciprocal matrix there is immediately suggested, as an analogue for (4) of the classical definition of fundamental sets,

DEFINITION 6. *A matrix of solutions $\theta(t)$ of* (4) *will be termed a fundamental set of solutions at $t=t_0$ in case*

$\theta(t_0)$ is a limited matrix, and in case (28)

there exists a unique limited reciprocal matrix for $\theta(t_0)$. (29)

In establishing the fact that a set of solutions, fundamental at t_0, is fundamental for all $|t|<r$, there will be needed

THEOREM IV. *Let $\xi(t)$ be the solution of* (4) *defined by the initial condition*

$$\xi(t_0)=\xi_0, \qquad (|t_0|<r). \tag{30}$$

Then the analytic continuation of $\xi(t)$ to the point $t=0$ can be obtained, and the expansion there converges for all $|t|<r$.

This result is obtained by recalling Proposition (3). The condition (30) defines a solution in the neighborhood of t_0, and this solution can then be continued to the point $t=0$ by successive applications of Proposition (3) and Theorem I.

THEOREM V. *If $\theta(t)$ is a fundamental set at $t=0$, and if ξ_0 is a point in Hilbert space, then there exists a point η of Hilbert space such that the solution $\xi(t)$ of* (4) *defined by the initial condition $\xi(0)=\xi_0$ is given by $\xi(t)=S\theta(t)\eta$.*

Let η be determined so that $S\theta(0)\eta=\xi_0$, where, since $\theta(0)$ has a unique reciprocal $\phi(0)$, it follows from Property (9) that $\eta=S\phi(0)\xi_0$. Consequently, there is obtained

$$\xi(t)=S\lambda(t)\xi_0=S\lambda(t)\,(S\theta(0)\eta)=S(S\lambda(t)\theta(0))\eta=S\theta(t)\eta,$$

where the interchange of summations is effected by means of Property (5).

The definition given for fundamental sets of solutions is seen to have content because the elements of a matrix of solutions $\theta(t)=(x_{ij}(t))$ can be defined by the initial conditions

$$x_{ij}(0)=d_{ij}, \qquad (d_{ii}=1;\ d_{ij}=0,\ i\neq j). \tag{31}$$

For $t=0$, $\theta(t)$ reduces to the unit matrix which is limited and possesses a unique limited reciprocal matrix. As a matter of fact, the matrix $\lambda(t)$ of Theorem I can easily be shown to be the matrix defined by the conditions (31).

With the aid of Theorem III consider the proof of the following theorem which states an important property of fundamental sets of solutions of (4).

THEOREM IV. *Let $\theta(t)$ be a matrix of solutions of* (4) *which is fundamental at $t=t_0$. Then, the analytic continuation of $\theta(t)$ to the point t_1 is fundamental at t_1 for all $|t_1|<r$.*

In the first place, it is easily seen that $\theta(t_1)$ is limited for all $|t_1|<r$, because $\theta(t)$ can be continued to the point $t=0$, and remains limited at every point reached in the continuation. Theorem II then shows that $\theta(t)$ is limited $L(0)l'(t)$ where $L(0)$ is the limit of the matrix $\theta(0)$. It remains to prove that $\theta(t_1)$ possesses a unique reciprocal matrix.

Let $\theta^{-1}_{(t_0)}$ be the reciprocal matrix of $\theta(t_0)$. Define a matrix $\phi(t)$ of solutions of the adjoint system (26) by the initial condition $\phi'(t_0)=\theta^{-1}_{(t_0)}$, where ϕ' represents the transposed matrix of ϕ. As a consequence of Theorem III it follows that in a sufficiently small neighborhood of t_0 the ϕ and θ satisfy

$$\frac{d}{dt}(S\phi'(t)\theta(t))=0.$$

Hence, in this neighborhood the product matrix $S\phi'(t)\theta(t)$ must remain constant and, therefore, is always equal to the unit matrix since this is the value it assumes for $t=t_0$. Since $\phi'(t)$ is limited, $\theta(t)$ therefore possesses a limited left-hand reciprocal $\theta^{-1}_{(t)}=\phi'(t)$ for all t in the restricted neighborhood of t_0. By analytic continuation of the matrices ϕ and θ this property is seen to hold for all $|t|<r$. It is easily seen that the existence of two distinct left-hand reciprocals of $\theta(t)$ for any value of t would imply a similar condition at $t=t_0$. Hence there exists a unique left-hand reciprocal $\theta^{-1}(t)=\phi'(t)$ for all $|t|<r$. By Property (8) of § 2 it follows that ϕ' is also the unique right-hand reciprocal of $\theta(t)$ for $|t|<r$.

§ 6. *The Non-Homogeneous System.*

Consider the infinite system

$$\frac{d\xi}{dt}=S\varkappa(t)\xi+\gamma(t), \tag{32}$$

where the function $\gamma(t)$ satisfies the conditions specified in hypothesis

(H_2). *The components $y_i(t)$ of $\gamma(t)$ are analytic in t and are regular for $|t|\leqq r$. Furthermore, the coefficients γ_n in the expansion for $\gamma(t)$,*

$$\gamma(t)=\sum_{n=0}^{\infty}\gamma_n t^n,$$

are points in Hilbert space for which $M\gamma_n\leqq b_n$ where $\Sigma_{n=0}^{\infty}b_n r^n$ exists.

It is easily verified that the solutions of the systems (32) and (4) are related in the fashion stated in

THEOREM V. *Let $\xi_1(t)$ and $\xi_2(t)$ be two solutions of* (32) *belonging to Hilbert space for a certain range of values $|t|\leqq r_1<r$. Then $\xi_3=\xi_2-\xi_1$ is a solution of the system* (4).

PROPOSITION 5. *The function $\gamma(t)$ belongs to Hilbert space for every value of $|t|\leqq r$, and $M\gamma(t)\leqq\Sigma_{n=0}^{\infty}b_n|t|^n$.*

To show this, recall Property (1); it is seen that, if α is a point in Hilbert space, then

$$|S_m\alpha\gamma(t)|=|\sum_{n=0}^{\infty}(S_m\alpha\gamma_n)t^n|\leqq M\alpha(\sum_{n=0}^{\infty}b_n|t|^n). \tag{33}$$

From (33) it follows that $\gamma(t)$ is a Hilbert point for $|t| \leqq r$ and that the modulus $M\gamma(t)$ satisfies the inequality of the proposition.

PROPOSITION 6. *The series*

$$M\gamma(t) = \sum_{i=1}^{\infty} \bar{y}_i(t) y_i(t) \tag{34}$$

converges uniformly for $|t| \leqq r$.

It is verified that

$$\sum_{i=h}^{k} \bar{y}_i(t) y_i(t) \leqq \sum_{n,\, m=0}^{\infty} |t|^{n+m} \Big|\sum_{i=h}^{k} \bar{y}_{in} y_{im}\Big|, \tag{35}$$

where $\gamma_n = (y_{1n}, y_{2n}, \ldots .)$. As a consequence of hypothesis (H_2) it is seen that the series on the right in (35) is dominated by the series

$$\sum_{n,\, m=0}^{\infty} r^{n+m} (b_n b_m).$$

A discussion similar to that used in the consideration of Proposition 2, § 3, then establishes the uniform convergence of (34).

PROPOSITION 7. *The series giving the components $y_i(t)$ of $\gamma(t)$ converge in t uniformly with respect to the index i.*

This result follows from the fact that the series for $y_i(t)$,

$$y_i(t) = \sum_{n=0}^{\infty} y_{in} t^n,$$

is dominated by $\Sigma_{n=0}^{\infty} b_n t^n$.

PROPOSITION 8. *Let the notation for the analytic continuation of $\gamma(t)$ to the point t_0 be*

$$\gamma(t) = \alpha(s) = \sum_{n=0}^{\infty} \alpha_n s^n, \qquad (s = t - t_0;\ |s| < r - |t_0|).$$

Then it follows that the α_n are points in Hilbert space with moduli $M\alpha_n \leqq a_n$, where $\Sigma_{n=0}^{\infty} a_n |s|^n$ converges for all $|s| < r - |t_0|$.

The proposition states that the analytic continuation of $\gamma(t)$ satisfies conditions similar to those imposed on γ in (H_2). The proof will not be given since it would be identical in method with that given for Proposition (3).

The existence theorem for system (32) will be established by a method analogous to the classical method of *variation of parameters* used for a finite system of differential equations. In the proof use will be made of the results obtained in the preceding articles with reference to fundamental sets of solutions and concerning the adjoint system. In view of Theorem V it is necessary to exhibit only one particular solution of (32). The existence of a solution taking on the initial value $\xi(0) = 0$ is established in

THEOREM VI. *Let hypotheses* (H_1) *and* (H_2) *be satisfied. Then there exists uniquely a function* $\xi(t)$ *of the class* $\mathfrak{M}$, *for* $|t|<r$, *satisfying the equation* (32) *and assuming the initial value* $\xi(0)=0$.

Let $\theta(t)$ be a fundamental set of solutions of the homogeneous system (4), and let $\phi(t)$ be the reciprocal of $\theta(t)$. Recall the analytic character of the matrix ϕ which follows from the fact that it constitutes a matrix of solutions of the adjoint system (26). Consider the infinite system of linear equations

$$\xi=S\theta(t)\eta \tag{36}$$

as giving a transformation of the variable ξ in (32) from ξ to η. Then

$$\frac{d\xi}{dt}=S\frac{d\theta(t)}{dt}\eta+S\theta(t)\frac{d\eta}{dt}, \tag{37}$$

provided that the term by term differentiation was legitimate. On substituting (36) and (37) in (32) there is obtained

$$S\frac{d\theta}{dt}\eta+S\theta\frac{d\eta}{dt}=S\varkappa(t)(S\theta(t)\eta)+\gamma(t)=S\{S\varkappa(t)\theta(t)\}\eta+\gamma(t), \tag{38}$$

where the interchange on the right is justified by Property (5) of § 2. Since

$$\frac{d\theta(t)}{dt}=S\varkappa(t)\theta(t),$$

equation (38) reduces to

$$S\theta(t)\frac{d\eta}{dt}=\gamma(t). \tag{39}$$

The equation (39) becomes, in view of Property (9) of § 2,

$$\frac{d\eta}{dt}=S\phi(t)\gamma(t). \tag{40}$$

The equation (40) has the solution

$$\eta(t)=\int_0^t S\phi(t)\gamma(t)dt, \tag{41}$$

where the integration may be taken along the straight line in the t-plane leading from $t=0$ to $t=t$. In case it can be established that the η of (41) belongs to Hilbert space for $|t|<r$, and is of such a nature that the term by term differentiation in (37) was legitimate, it then follows that $\xi(t)=S\theta(t)\eta(t)$ is a solution of (32). To complete the proof it would remain to show that $\eta(t)$ is a function of the class $\mathfrak{M}$.

Let Property (2) of § 2 be applied to $S\phi(t)\gamma(t)$. Every row of $\phi(t)$ is a solution of the adjoint system (26) and hence, in view of Corollary (1) to Theorem I as applied to (26), the modulus of each row is uniformly bounded for $|t|\leqq r_1<r$. Therefore, since $S\bar{\gamma}(t)\gamma(t)$ converges uniformly, it follows that $S\phi(t)\gamma(t)$ converges uniformly, and hence represents an analytic function.

Consequently, every coordinate of $\eta(t)$ is an analytic function regular for $|t|<r$. Moreover, if α is a point in Hilbert space,

$$|S_m\alpha\eta(t)|\leqq\left|\int_0^t S_m\alpha\{S\phi(t)\gamma(t)\}dt\right|\leqq M\alpha\left|\int_0^t M\gamma(t)L(t)dt\right|,$$

where $L(t)$ is the limit of the matrix $\phi(t)$. As a result of Property (1) of § 2 it follows that $\eta(t)$ is a Hilbert point for $|t|<r$, and its modulus is uniformly bounded for $|t|\leqq r_1(r_1<r)$ in view of the similar property which holds for $M\gamma(t)$ and $L(t)$.

To justify the equation (37) for the $\eta(t)$ of (41) it must be shown that the two series on the right in (37) converge uniformly. From (40) it is seen that the modulus of $d\eta/dt$ is at most $M\gamma(t)L(t)$ which is uniformly bounded for $|t|\leqq r_1<r$. As a result of Corollary (2) to Theorem I, the two infinite series

$$S\left(\frac{\overline{d\theta'(t)}}{dt}\right)\left(\frac{d\theta(t)}{dt}\right),\ S\bar{\theta}'(t)\theta(t),$$

converge uniformly since every column of θ is a solution of (4). Hence it is seen that Property (2) of § 2 may be applied to each of the series in (37) to establish their uniform convergence.

On substituting (41) in (36) it follows that

$$\xi(t)=S\theta(t)\eta(t)=S\theta(t)\left[\textstyle\int_0^t S\phi(t)\gamma(t)dt\right] \tag{42}$$

is a solution of (32) which satisfies $\xi(0)=0$. Since $M\eta(t)$ was uniformly bounded for $|t|\leqq r_1<r$ it follows that $M\xi(t)$ will also be uniformly bounded. It then is seen that, as a consequence of Proposition (4), the function $\xi(t)$ belongs to the class $\mathfrak{M}$.

The solution given by (42) is the only solution of (32) for which $\xi(0)=0$ because, if there were another solution $\xi_1\neq\xi$, it would follow that the difference $\xi_2=\xi-\xi_1$ would be a solution of (4) for which $\xi_2(0)=0$, and therefore $\xi_2(t)=0$ for $|t|<r$.

The existence of a solution of (32) could also be established directly by the method of undetermined coefficients, but the proof of the result is much more difficult than the corresponding proof for the system (4).

Various problems are suggested in connection with systems (4) and (32) by the many known properties of finite systems of linear differential equations. Moreover, systems in which the variables are real and the coefficients are real and continuous may be considered under conditions analogous to those of (H_1) and (H_2).

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